Introductory note by Translator:

If not already, the reader of the *Principia* needs to be aware of Newton's method of presenting material : first a proposition is stated succinctly, then it is elaborated on, in an extended summary of the proposition, without giving a detailed account of the steps involved in its solution, and finally, a discourse is presented on how the proposition may be shown, often building on previous propositions, lemmas, etc., usually presented geometrically. These geometric interpretations may be unsatisfactory, especially if a curve is required to be plotted, and the initial method may have involved some form of calculus which may not be revealed or simply sketched out, by which Newton has proven the results for himself, except perhaps for salient results found, that may be quoted in isolation. A critic may advise that there is no surviving evidence for Newton's calculus being involved in the *Principia*, except what he states as such in Lemma II, Section 2 of Book 2, and the explicit use made in some sections, such as the derivation of the cylcoid properties in Book I, Section 10. If fact. two kinds of limiting processes have been considered for finding the rate of change of a quantity at a point, the one algebraic or analytical, and the other geometrical; in the first case a variable quantity such as $A$ in the relevant formula has a small 'moment' $a$ added to it (sometimes the moment is called $o$), the difference of the formulas involving $A + a$ and $A$ is evaluated, and the resulting difference (called the *genitam* by Newton) is divided by $a$ , in which case the rate of change of the generating formula with respect to $A$ is found as a finite ratio at the point $A$; and in the latter case, for example, the ratio of two merging line segments forms a limiting value associated with other finite ratios, this approach is used for example in Section X of Book I on the cycloid. Whatever the beginnings of this work, we have to accept it as it is; everything else of a hidden nature must remain so, as no documentary evidence exists to support a particular viewpoint, other than what has been examined already.

The effect of the *Principia* was a wake–up call for the rest of humanity, which had slumbered blissfully in dogma for millennia – a thunderbolt out of the blue preceded by a few distant rumbles; the question had to be asked : How on earth did he manage to do all this? A terrible fate had fallen as the lot of the more observant part of humanity : *Newton was asking people to start thinking for themselves*.....For though he outlined what he had done, as we have just noted, the details were often scanty on how exactly he had brought it about.

Hence extra notes are required to be written into this translation, or derived from other sources such as the LeSeur & Janquier translation of 1737, and these should come at the end of Newton's treatment of the proposition, which we call sometimes : Note *A*, though sometimes it seems better to add shorter comments as one proceeds. Brougham's & Routh's *Newton's Principia* give essentially the modern explanation of some but not all of the propositions demonstrated in this section, and in the following two – the exponential function and indeed the idea of a function as we understand it , was not available when Newton worked out these sections – which he did using geometric and arithmetic progressions only, to discuss the solutions of these problems involving logarithms – and these notes are tacked on at the end also, and of course can be worked out without any trouble, and which we call sometimes : Note *B*. However, the idea in this translation generally is to get back to something approximating the foundations of Newton's original calculations, while still making the book accessible to us. Thus we begin with some of the extra material added by L & J, which as they admit somewhere else, relies on the genius of Euler at times :

Added Note (32):

The Leseur & Janquier definition of the Logarithmic Curve and related matters.
[The reader will recognise this as the antilogarithmic or exponential curve, here treated geometrically rather than analytically.]

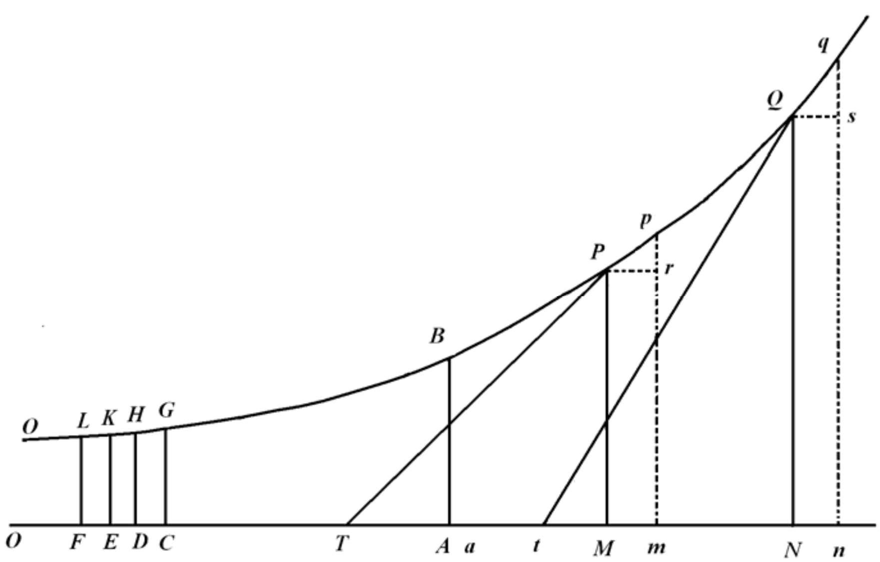
Along that right line NAO the perpendicular *MP* may be carried in a uniform motion and parallel to itself, while on that line of the perpendicular *MP* , the moveable point *P* may be moving with a variable velocity according to this law: that the velocity of this point shall always be proportional to its distance from the line *NAO*, and the curve described by that point P may be called logarithmic or logistic. [Thus, the rate of change of this quantity is equal to itself.]

The line *NAO*, that *PM* follows with a uniform motion parallel to itself, may be called the axis of the logarithmic, and the lines *PM*, *QN* perpendicular to the axis are its ordinates.

If a certain line from the ordinates of the logarithmic, as *AB*, shall be equal to unity, the axis point *A* put in place may be agreed to be called the origin of the abscissae, and the ordinates taken from the part *AM* are positive, from the part *AO* negative, and the abscissae belonging to the ordinate *AB* or to unity is itself 0.

*Corol.* 1 *The minimum differences of the logarithmic ordinates arising in equal times are as these ordinates*.

For indeed the velocity by which the ordinates may increase or decrease at some point of the logarithmic perpendicular to the axis, is proportional to the ordinate, from the



definition, but during an infinitely small time interval that velocity may be agreed to be constant, and in equal increments of time the increments of the lines either increasing or decreasing are as the uniform velocities by which they are generated, therefore the increments or decrements of the ordinates, that is, the difference of these in equal increments of time, are as these ordinates.
[Thus, by definition, the speed $v$ of the point $P$ is proportional to the length of the ordinate $y$, or $v \propto y$ ; again, the rate of increase of the speed $v$ is proportional to $v$, so that the acceleration $a = \frac{dv}{dt} \propto v = \frac{dy}{dt} \propto y$ in turn.]

*Corol.* 2. *Let PM, QN be whatever ordinates, and two other ordinates pm, qn may be drawn as close as possible to these, equally distant from these: pm and qn shall be proportional to the initial ordinates.*

For the velocity by which the ordinate may be carried along parallel to itself is uniform, and thus in the same time the ordinate *PM* may arrive at *pm*, and *QN* at *qn*, on account of the equal distances; therefore by Corol. 1, the differences of the ordinates while they arrive at *pm* and *qn* shall be proportional to these ordinates, but with the differences of these lines added or subtracted from the lines *PM* and *QN* , they become the ordinates *pm*, *qn;* and the initial ratio shall not be changed with the corresponding terms in a ratio equal to themselves, with the terms of whatever ratio added or subtracted. Therefore the ordinates *pm* and *qn* shall be to each other as *PM* to *QN*, and also on alternating, $PM : pm = QN : qn$ .

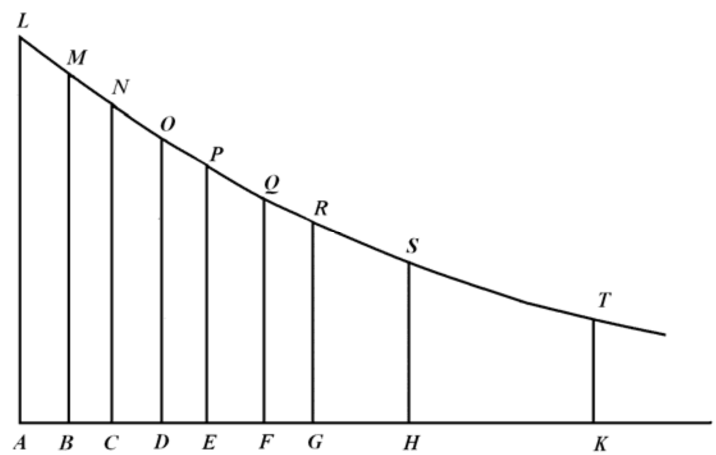
*Corol.* 3. *If the points C, D, E, F may be taken on the axis at equal and minimal distances, the ordinates may be erected at these points, then these ordinates constitute a geometric progression.*

For since by hypothesis the ordinates *GC* and *HD, HD* and *KE* are minimally and equally distant, by the preceding corollary there is $GC : HD = HD : KE$ , and in the same ratio there is $HD : KE = KE : LF$ , and thus henceforth, from which it is clear that the ordinates $GC : HD : KE : LF$ ,etc. are in a geometric progression.

33. *Theorem* I. *Four points may be taken on the logarithmic axis , thus so that the first two may be mutually equally distant from the second two, then the ordinates erected from these points are in geometric progression. And if any points may be taken in a continuous order on the axis, then the ordinates put in place from these will be in geometric progression.*

Any two points may $A$ and $E$ may be taken on the axis, and some other two $H$ and $K$ such that $AE = EH = HK$, and the ordinates *AL, EP, HS*, and *KT* may be erected at these points; I say that these points are in geometric progression. For AE as well as HK may be divided into an infinite number of parts equal amongst themselves, and there will be just as many divisions in each interval; ordinates may be erected at these points that become two geometric progressions, in which there are just as many terms, and the ratio of the successive terms will be equal, because the ordinates in each progression are equally distant; therefore from the equality, the first term *AL* of the first progression will be to *EP* the final term of this progression, as *HS* the first term of the other progression to the final term of that *KT*. Q.e.d.

And if many points may be taken equally distant on the axis with the order continually succeeding amongst themselves, the order in these points erected will be in a geometric




progression: as definitively approved in Cor. 3.

*Cor. From the converse, if on some line a number of points are taken, with the ordinate equally distant, and on these perpendiculars are erected which shall be in geometric progression, some logarithmic curve will pass through the extremities of these perpendiculars*.

For let *A, D, G*, etc be these equally distant points and the intervals of these may be divided into as small as possible equal parts, there will be just as many in any interval, just as many mean proportionals may be assumed between the perpendiculars *AL* and *DO, DO* and *GR*, etc. as there are points of division, and at the individual points perpendiculars may be erected with these mean proportionals taken in order ; so that the curve may touch the given perpendiculars *AL, DO, GR* as well as these means, I say that the curve is logarithmic.

For it is readily apparent from the nature of the progression that since there shall be $AL : DO = DO : GR,$etc, and just as many mean proportionals may be assumed between *AL* and *DO*, as the number assumed between *DO* and *GR*, and thus henceforth, a continued constant progression to be formed from all these mean proportionals from the given as well as from the found, thus any from these, such as *AL*, to be to the nearest to itself *BM*, as some other *DO* is to that nearest *PE*, from which on separating the ratio, *AL* shall be to its difference from the nearby term, as also *DO* is to its difference from the nearby term, and thus the differences of the nearby perpendiculars will be everywhere proportional to these differences; Therefore with the vanishing of the points taken with the intervals on the axis, and with the perpendiculars everywhere equal to the speed of the side and equal to the time increment, the velocities by which the perpendiculars increase or decrease will be proportional to these perpendiculars; Therefore, from the definition of logarithms, that curve which may touch these perpendiculars will be logarithmic.

34. *Theorem* II. *The abscissae of the logarithmic axis are the logarithms of the ordinates put in place at the end of these.*

[This theorem deals with the logarithmic or exponential curve of the form $y = e^{x}$, and its inverse function $x = ln\, y$, as understood at the middle of the 18th century.]

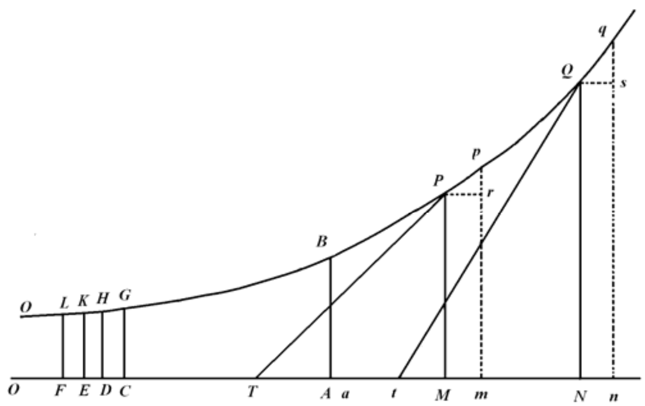
Hence from the origin of the axis minimal equal parts, at the ends of the individual ordinates, all these ordinates constitute a geometric progression among the terms of which there occurs unity, truly the abscissae of these points will be in an arithmetic progression on account of the equality of the parts assumed on the axis, and the abscissa which corresponds to unity is 0; But now with the terms of the arithmetic progression prepared, among which is 0, thus with the terms of a geometric progression so that 0 may correspond to unity and the remaining terms themselves may correspond, then the terms of the arithmetic progression are the logarithms of the corresponding terms of the geometric progression; Therefore the logarithmic abscissae are the logarithms of the corresponding ordinates.

*Corol.* 1. *The portion of the axis which is placed between two ordinates is the logarithm of the ratio which lies between these two ordinates.*

For the quotient of the two quantities expresses the ratio which lies between these, and the difference of the logarithms of these quantities , is the logarithm of the quotient of these, but the abscissae are the logarithms of the ordinates, and the portion of the axis which is intercepted between two ordinates is the difference of the abscissas or of the logarithms pertaining to these ordinates; therefore that portion is the logarithm of a quantity which expresses the ratio which comes between the ordinates.

*Corol.* 2. *If the logarithms of two or more quantities may be given, and from a given point of some line, lengths are taken equal to these logarithms, and at the ends of these lengths perpendicular quantities may be erected the logarithms of which may be taken equal [to the logarithms given], the logarithmic curve will pass through the ends of these perpendiculars.*

On the right line *OAN* there may be taken a point *A* at which the perpendicular *AB* equal to unity may be erected, and let *AM* be the logarithm of the quantity which is equal to the perpendicular *MP*; *A* shall be from the difference of the terms of the arithmetical progression from which the logarithms are taken, which thus is accurately contained in the interval AM, just as often as that number of terms are contained in the geometric progression from which quantities are taken of which the logarithms may be found. Indeed, just as many mean proportionals may be sought between the lines *AB* and *MP* which are the points of division between *A* and *M*, and at these points perpendiculars may be erected equal to the mean proportionals of these ordinates, from which a geometric progression may arise, which is that progression itself of the quantities the abscissae of which of the line *OAN* are logarithms from the quantity *A* onwards by successive increases, if indeed in each progression the terms *AB* and *MP* occur with the same interval in each separate interval, but if at equidistant points some perpendicular lines may be erected in a geometric progression, some logarithmic curve will touch the vertices of these, by Cor. Theorem 1. Therefore if numbers may be given, always to be considered with their logarithms, there will be the logarithmic curve of which the abscissae shall be these logarithms, and the ordinates of which shall be the corresponding quantities.




[We may note the hallmark of the exponential function [and of such functions and their inverses in general] to arise from these associated arithmetic and geometric means and hence progressions : if

$$y_1 = e^{x_1} \text{ and } y_2 = e^{x_2} \text{ then } \sqrt{y_1 y_2} = e^{\left(\frac{x_1+x_2}{2}\right)},$$

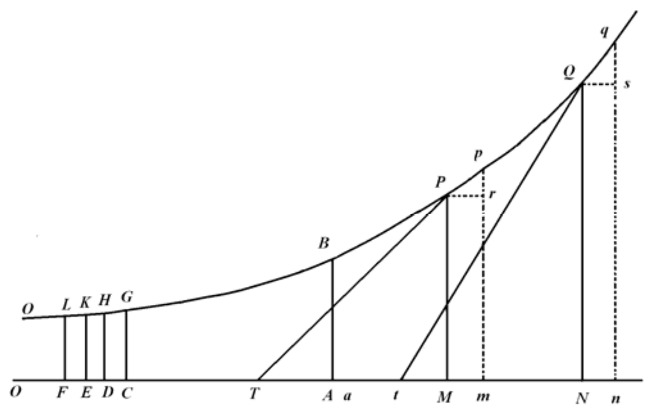
and conversely, if $x_1 = ln\, y_1$ and $x_1 = ln\, y_1$ then $\frac{x_1+x_2}{2} = ln\sqrt{y_1 y_2}$.]

[We note the following theorems briefly, until we come to one of some relevance: ]

35. *Theorem* III. *The axis of the logarithmic is the asymptote of this to which from one part it approaches closer to some given quantity yet never reaches that, and from the other part it recedes further from some given quantity*.

36. *Theorem* IV. *The subtangent of the logarithmic curve is constant*.

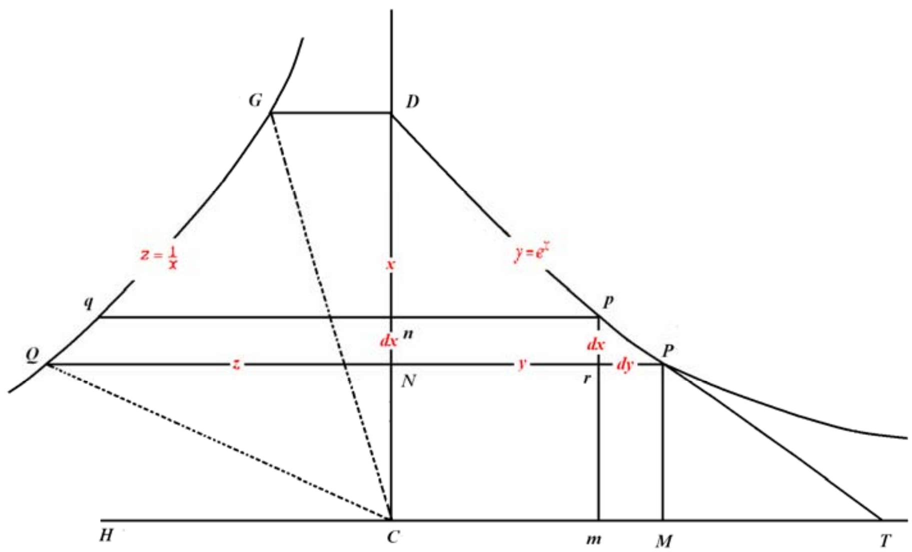
For there may be taken anywhere on the axis the smallest equal parts *Mm*, *Nn*, and with the ordinates erected *MP*, *mp*, and *NQ*, *nq*, through the points *P* and *Q* the tangents crossing the axis at *T*, *t* may be considered; also the right lines *Pr*, *Qs*, may be drawn to the perpendicular ordinates *mp, nq* . With the vanishing of the ordinates with the distances *Mm*, *Nn*, the triangle *Ppr* shall be made similar to triangle *TPM*, and triangle *Qqs* becomes similar to *tQN*, and thus we have :




$pr:PM = Pr(\text{or } Mm):MT$ and $qs:QN = Nn(\text{or } Mm):Nt$, but on account of the equal distances *Mm, Nn* , there is $pm:PM = qn:QN$ , and on separating there becomes $pr:PM = qs:QN$ whereby $Pr(\text{or } Mm):MT = Nn(\text{or } Mm):Nt$, and thus $MT = Nt$ . Q.e.d. Hence calling *x* the abscissa *AM* , *y* the ordinate *MP*, *s* the subtangent *MT*, the fluxion *Mm* will be *dx*, $pr = dy$ , and since there shall be $\frac{y}{s} = \frac{dy}{dx}$ or $ydx = sdy$ .

[We may note also (Theorem V), that there are different kinds of logarithmic curves, with differing subtangents. There now follows a number of problems relating to these theorems. However, we leave these and pass on to : ]

44. *Theorem* VI. *The hyperbola QqG shall be described with orthogonal asymptotes CH, CD, and through the given point D on the asymptote CD, the logarithmic curve DpP shall be drawn having the axis CH produced; through the point D to the ordinate of the hyperbola DG , and through some other point N with the ordinate NQ which produced may cross the logarithmic curve at P; the hyperbolic area NQGD will be to the power of the hyperbola, or to the rectangle* $CD \times DG$ *, in the ratio of the right line NP to the subtangent of the logarithmic curve.*



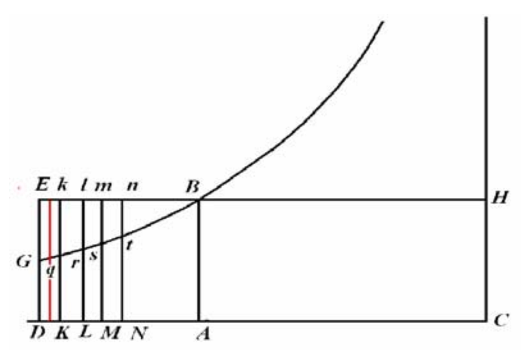

For another line *qp* may be drawn infinitely close to *QP* itself, and from the point *p,* a perpendicular *pm* may be sent to the axis *CT* cutting *QP* in *r*, and *PM* sent likewise. The line *PT* may touch the logarithmic curve at *P*; on account of the similar triangles *prP*, *PMT* there is $pr\ (\text{or } Nn) : Pr = PM\ (\text{or } CN) : MT$, and (on account of the nature of the hyperbola by Theorem 4 of Book I) $NQ : DG = CD : CN$ ; and thus on combining the ratios and from the equation, $NQ \times Nn : Pr \times DG = CD : MT$ .
[Thus,
$\frac{Nn}{Pr} = \frac{CN}{MT}$ , and $\frac{NQ}{DG} = \frac{CD}{CN}$ ; hence
$\frac{Nn \times NQ}{Pr \times DG} = \frac{CD}{MT}$ or $Nn \times NQ = Pr \times \frac{DG \times CD}{MT}$.]
Whereby on account of *CD* and *MT* given, the sum of all the rectangles $NQ \times Nn$ , into which it is possible to divide the area *NQGD,* that is, this area itself of the hyperbola is to the rectangle under the given *GD*, and the sum of all *Pr*, [recall that the subtangent *MT* has a constant length for all the values considered] or to the whole right line *NP*, is as *CD* to *MT,* and hence $NQGD \times MT = NP \times GD \times CD$ , and hence
$NQGD : GD \times CD = NP : MT$. Q.e.d.
[To complete the proof, we need to apply a transformation and make *y* the independent variable for the right-hand curve: that is consider the inverse function $x = ln\, y$ , then the

limits of the integration on the right become $x_1 = ln\, y_1$ and $x_2 = ln\, y_2$, where *NP* or *y* becomes $ln\frac{y_2}{y_1}$ the difference of the logarithms corresponding to points on the common axis *CD* , *i.e.,* on the *x*-axis of hyperbolic curve. This has been demonstrated above in Cor. I, Theorem II]

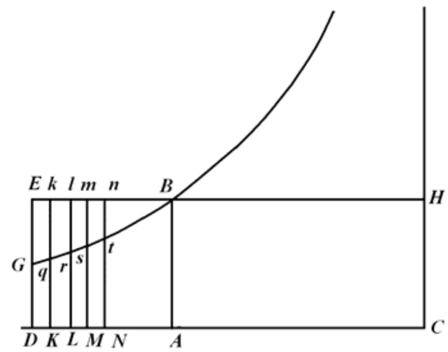
[Leseur & Janquier Extended Note on PROPOSITION III. PROBLEM I. :

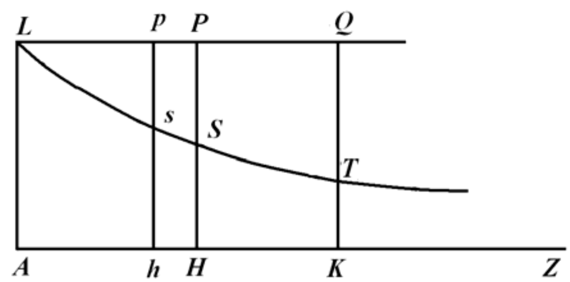
For in the ascending case the rectangle *DB* may be resolved into innumerable rectangles *Dk, Kl, Lm, Mn,* etc., which shall be as decrements of the velocities made in just as many equal increments in the times, and they will be, zero, *Dk, Dl, Dm, Dn*, etc, as the whole velocities sent off at the beginning of the individual equal times. Therefore because the whole rectangle *DB* shows the initial ascent, the velocity of the body and the resistance of the medium is proportional to the velocity, the rectangles *AE, Ak, Al, Am, An*, etc. show the remaining velocities, and the resistances of the medium from the beginning of the individual equal times. Make *AC* to *AK*, or the rect. *AH* to the rect. *Ak*, as the force of gravity to the resistance from the beginning of the second time, and the resistance is added to the force of gravity – because gravity and the resistance of the ascending body retard the ascent, and there will be *DEHC, KkHC, LlHC, MmHC,* etc., as the absolute forces by which the body is retarded at the beginning of the individual times, and thus as the decrease in the velocity, that is, as the rectangles *Dk, Kl, Lm, Mn*, etc, and therefore by Lemma I of this book, in a geometric progression. Whereby if the right lines *Kk, Ll, Mm, Nn,* etc., may meet the hyperbola in *q, r, s, t,* etc., the areas *DGqk, kqrL, lrsM*, *MstN*, etc will be equal, and thus both the times as well as the forces of gravity are always in equal proportions.

A perpendicular may be erected in the middle of the part *DK* as far as *EB*, the area *DGqK* will be to the area *GEkq* as the part of this perpendicular to the hyperbolic ordinate to the remaining part of this as far as to *EB*; but from the properties of the hyperbola, that ordinate to the hyperbola is as *AB* or to the total perpendicular, as AC to the abscissa of this ordinate, and thus on separating the ratio, this is the ordinate to the perpendicular part remaining as far as to the line *EB*, or the area *DGqK* is to the area *GEkq* as *AC* to the part of the abscissa between *A* and the perpendicular, and with the common altitude *AB* assumed, as the rectangle *AH* to the rectangle under AB and the portion of the abscissa between *A* and the perpendicular, and thus the area *DGqK* to the area *GEkq* as the force of gravity to the resistance or remaining velocity in the middle of the first time, and since the force of gravity shall be the same everywhere and the area *DGqK, qKLr,* equal everywhere, the areas *GEkq, kqrl*, etc., shall be always as the resistances at the individual times or as the velocities, and thus as the individual distances described in the individual times, and as a consequence the whole areas *GEnt* , will be as the distances described in the whole times, while the areas *ABNn* will be as the velocities remaining at the end of these times.

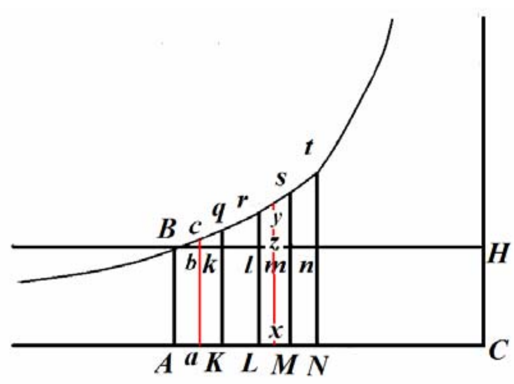


If some logarithmic curve *LST* shall be described with the asymptote *AZ*, the asymptote approaching towards Z, and the velocity ordinate *AL* shows the initial motion of the body, and the abscissae *AH*, *AK* show the times; the ordinates will be *HS*, *KT*, as the remaining speeds in the lapsed times *AH*, *AK*, and thus with the line *LQ* drawn through the point *L* parallel to the asymptote *AZ*, and the ordinates produced *HS*, *KT* on cutting at *P*, *Q* ; *PS*, *QT* are as the velocities removed, and also as the distances described, in the times *AH*, *AK* or *LP*, *LQ*. With the ordinate *hs* drawn, infinitely close to the other *HS*, the distance described with a uniform speed *AL*, in the element of time *hH* in free space, will be to the distance in the same time with the velocity *HS*, accomplished in the resisting medium, as the rectangle $HP \times Hh$ to the rectangle $SH \times Hh$, or the area *HSsh*, and thus if the total time *AH* may be divided into innumerable small parts as *hH* shall be divided, the distance described in the whole time *AH* in a vacuum will be, to the distance traversed in the same time in a resisting medium as the rectangle *AP* to the logarithmic area *ALSH*; but the area *ALSH*, is equal to the rectangle of the logarithmic subtangent in *PS*, and thus if *AL* shall be assumed equal to the subtangent, the area *ALSH*, is equal to the rectangle $AL \times PS$. Whereby in this hypothesis, the first distance will be to the second as *LP* to *PS*.




Indeed for the falling body, the velocity acquired in the given time, *ABrL*, to the velocity acquired in some other time *ABtN*, as the rectangle *Al* to the rectangle *An*, or as the given line *AL*, to the line *AN*, from the demonstration, and thus the velocity of the falling body with the area *ABtN*, or with the time continually inceasing. But with the point *N* coinciding with the point *N* and with the asymptote *CH*, the area *ABtN* becomes infinite, that is, the time becomes infinite and the velocity a maximum; Whereby the maximum velocity which also may be called terminal, is to the given velocity acquired at some time *ABrL*, as *AC* to *AL*, or as the rectangle *AH*, to the rectangle *Al*, that is, as the force of gravity to the resistive force at the end of the time *ABrL*.



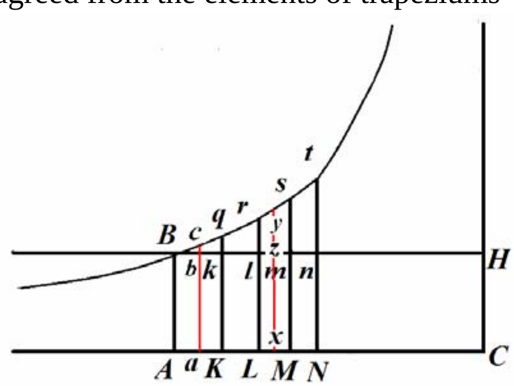

In the ascent of the body in the times *DGqK*, *DGrL*, *DGsM*, etc., increasing in an arithmetic progression, the abscissae *CD*, *CK*, *CL*, etc., decrease in a geometric progression, but the individual themselves, from the demonstration, are as the sum of the maximum velocity that the line *CA* sets out, and the remaining velocities that the lines *AK*, *AL*, or *AM*, etc., set out at the end of the times *DGqK, DGrL*, or *DGsM*, etc. Whereby with the time increasing in an arithmetic progression, the sum of the maximum velocity and of the remaining velocity in ascending decreases in a geometric progression. In a similar manner in the descent of the body it is apparent that with the increases in the times – as shown in the figure – *ABqK, ABrL, abSM*, etc., in an arithmetic progression, the abscissae *CA*, *CK*, *CL*, etc., decrease in a geometric progression, but these abscissae are as the differences of the maximum velocitiy that the line AC exhibits and of the velocity acquired that the lines AK, AL, AM, etc show; therefore with the increase in time in the arithmetic progression, the differences of the maximum velocities, and of the velocity acquired at some given time in the descent, decreases in a geometric progression. Hence if these sums in ascending and differences in descending are expressed by numbers, the times will be as the logarithms of these numbers.

For if in the ascent of the body the times *DGqK, KqrL, LrsM, MstN*, etc. may be taken equal, the distance described in the first time will be as $GEkq = DK \times DE - DGqK$; the distance described in the second time as

$qkrl = KL \times DE - KqrL($, or because $KqrL = DGqK) = KL \times DE - DGqK$ , and thus concerning the remainder. Whereby the differences of the distances described in the first and second times is as $DK \times DE - KL \times DE$ , that is, on account of *DE* given, as $DK - KL$ ; and by a similar argument the differences of the second and third times is as $KL - LM$ ; the difference of the third and fourth times as $LM - MN$ . Therefore the differences of the distances which are described in equal time differences will be as the differences $DK - KL$ , $KL - LM$ , $KL - LM$ , etc., but the terms *DK, KL, LM, MN*, etc, decrease as the terms of the geometric progression *DC, KC, LC, MC,* etc. Therefore the differences $DK - KL$ , $KL - LM$ , $KL - LM$ , etc., decrease as *DK, KL, LM, MN*, etc., or as the terms of the geometric progression *DC, KC, LC, MC*, etc.

Note on Corollaries to Proposition IIII, by Leseur & Jacquier : We may draw a red normal *abc* at the mid-point of *AK*, for since by that Lemma these areas can be agreed to be taken for straight lines, and a perpendicular *ac* may be erected in the middle of the part *AK* as far as the hyperbola, and it will be easily agreed from the elements of trapeziums that *ABqK* to be to the triangle *Bkq* as that whole perpendicular *ac* (for which *Kq* will be taken) to the portion *bc* of this understood to be within the triangle, which will be $\frac{1}{2}kq$ according to Euclid ; truly from the nature of the hyperbola [*i.e.* $ac \times aC = AB \times AC$ ] that perpendicular *ac* is to *AB*, as *AC* to *Ca* or $AC - \frac{1}{2}AK$

[*i.e.* $\frac{ac}{AB} = \frac{AC}{Ca} = \frac{AC}{AC-\frac{1}{2}AK}$ ] and on separating,




that perpendicular *ac* to *ac* – *ab* or *bc* which is $\frac{1}{2}kq$ as *AC* to $AC - AC + \frac{1}{2}AK$ or $\frac{1}{2}AK$ ;

[*i.e.* $\frac{ac}{ac-ab} = \frac{ac}{bc} = \frac{AC}{AC-Ca} = \frac{AC}{AC+\frac{1}{2}AK-AC} = \frac{AC}{\frac{1}{2}AK}$ ];

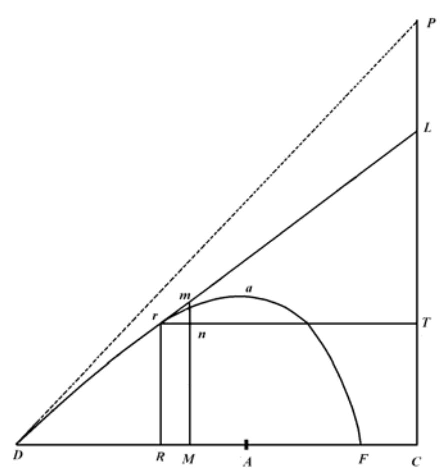
Hence the area *ABqK* is to the area *Bqk* as *AC* to $\frac{1}{2}AK$, or as the rectangle *ABCH* to the rectangle $\frac{1}{2}ABkK$, or as the force of gravity that the rectangle *AH* expresses to the resistance at the midpoint of the first time that the rectangle *Ak* expresses, since indeed *AK* shall be as the whole velocity acquired in the first time, $\frac{1}{2}AK$ will be as the velocity at the mid point of the first time; moreover the resistances are in proportion to the velocities.

Now with these same areas taken as rectilinear trapeziums: perpendiculars *xzy* may be drawn in the centres of the parts *AK*, *KL*, *LM*, *MN* as far as to the hyperbola, and from Euclid, it will be readily agreed that the whole area of the trapeziums, such as *rLMs*, is to the area of that part above *BH* – such as *rlms*, as the whole line *xy* drawn through the centre of the trapezium to the part *xy* above *BH*, but from the nature of the hyperbola that perpendicular *xy* is to the *AB* or *xz*, as *AC* to the abscissa *Cx* corresponding to that perpendicular (which is $CL-\frac{1}{2}LM$ ), and on separating that perpendicular *xy* is to the part *zy* of this above *BH*, as *AC* to *Ax*, the part of the abscissa between *A* and that perpendicular (that is, in the example assumed, as *AC* to $AL+\frac{1}{2}\times LM$ ). Hence the total area of the individual trapezium to the area of this part above *BH*, as *AC* to *Ax* the part of the abscissa between *A* and the middle of this part assumed, or (with a common altitude assumed *AB*) as the rectangle *AH*, to the rectangle under *AB* and the line between *A* and the middle of the part assumed to be taken; but the one is as the force of gravity, the other as the velocity and therefore as the resistance in the middle of the time that corresponds to that assumed part, hence alternatively, the area of the individual trapezium is to the force of gravity as the part of the trapezium above *BH* to the resistance or to the velocity in the middle of the time to which the trapezium may correspond; but the areas of the trapeziums are equal everywhere, and the force of gravity always the same, therefore the ratio of these is constant; therefore, the parts of the trapeziums above *BH*, as *rlms* are thus as if resistances or velocities, and thus they correspond to the distances described in the individual elements of time.

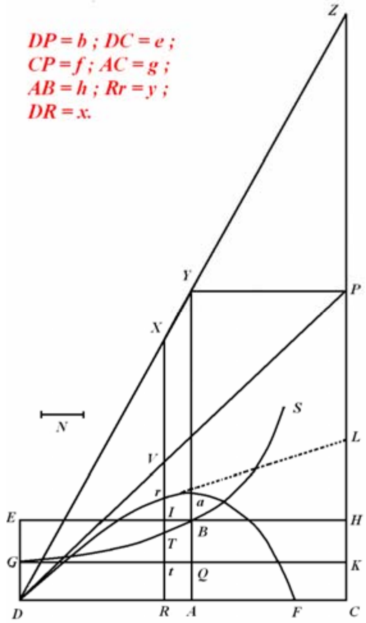
Extended Note *A* from L & J on Proposition IV:

Note 54:

But since *DA* is to *AC* as the resistance from the vertical motion arising from the start to the force of gravity, the total time of the ascent of the body will be *DABG* (by Prop. III of this work : see the above diagram), in which time also the body will have traversed the length *DA* horizontally, and thus when it will have arrived at its own maximum height it will be on the perpendicular *ABa*, and always afterwards it will approach towards the asymptote *PC* (by Cor. Prop. II). Through some point of the trajectory *r*, (see diagram opposite) *rT* is drawn parallel to the horizontal *DC* and crossing the vertical *CP* in *T*, the vertical *Mm* infinitely



close to *Rr* itself will cut *rT* in *n* and the tangent *rL* of the curve in *m* : and since the motion of the body at the place *r* along the arc *rm* can be divided into the horizontal motion *rn* and the vertical *nm*, the horizontal velocity will be to the vertical velocity as *rn* to *nm*. But on account of the similar triangles *rnm*, *rTL*, there shall be $\frac{rn}{mn} = \frac{rT \text{ or } RC}{TL}$, and $\frac{rn}{rm} = \frac{RC}{rL}$. Whereby since *RC* shall be as the horizontal velocity of the body at the place *r* remaining from the velocity *DC* that it may have from the initial motion at *D* (per Cor. Prop. II); TL will be as the vertical velocity of the body left from the initial velocity *CP*, and *rL* as the oblique velocity on the arc *rm* from the two velocities compounded *rT* and *TL*. And thus the velocity and hence the resistance of the body at some point of the trajectory *r* is as the tangent to the curve *rL*.




Note 55: Hence through a given point of the trajectory *r* it is possible to draw the tangent *rL*. For the vertical velocity *LT* at the position *r* is to the vertical velocity *CP* at the position *D*, as the rectangle *RB* to the rectangle *DB* – see the above figure – or as *RA* to *DA* (by Prop. II); and thus $LT = \frac{CP \times RA}{DA}$.

Note 56: From the above construction the equation for the trajectory *DraF* is easily deduced. For on putting $\begin{matrix} DP = b, DC = e, CP = f, AC = g, \\ AB = h, Rr = y, \text{ and } DR = x, \end{matrix}$ there will be (by Theorem 4 *de Hyperbola*, Book I *Apollonius*) $\frac{DC(e)}{AC(g)} = \frac{AB(h)}{GD}; GD = \frac{gh}{e}$, and $\frac{RC(e-x)}{AC(g)} = \frac{AB(h)}{RT}$ and $RT = \frac{gh}{e-x}$, and thus $QB = AB - GD = \frac{eh-gh}{e}$, and the element of the hyperbolic area arising *RDGT* $RT \times dx = \frac{ghdx}{e-x}$, and hence the area $RDGT = gh \times \int \frac{dx}{e-x}$. In addition, from the construction, since $\frac{N}{QB} = \frac{DC}{CP}$,

$\frac{CP(f)}{DC(e)} = QB \frac{(eh-gh)}{e} : N$ ; $N = \frac{eh-gh}{f}$,

and, as $\frac{GTIE}{N} = RV - Vr = Rr$,

$Rr = y = \frac{DR \times AB - RDGT}{N}$.

Also, $DR \times AB = hx$. Whereby there will be $y = \frac{fx}{e-g} - \frac{fg}{e-g} \times \int \frac{dx}{e-x}$. Also from the construction, $\frac{DA}{AC}$ or $\frac{e-g}{AC}$ is as the resistance of the medium in the vertical motion to the force of gravity : [ $\frac{DA}{AC}$ or $\frac{e-g}{g} = \frac{kv_0 \sin\alpha}{g} = \frac{v_0 \sin\alpha}{g/k} = \frac{v_0 \sin\alpha}{v_{term.}} = \frac{f}{a}$ ] , and thus by Cor. I Prop. III, as the vertical velocity, that is shown by the right line *CP* or *f*, to the terminal velocity; and thus if the terminal velocity may be shown by the line *a*, there will be had

$a = \frac{fg}{e-g}$. Thence there becomes $y = \frac{ax}{g} - a\int \frac{dx}{e-x}$ and with the fluxions assumed (differentials) $dy = \frac{adx}{g} - \frac{adx}{e-x}$. If $RC$ or $e - x = z$ is put in place , there will be $-dx = dz$ , and $-\frac{adx}{e-x} = \frac{adz}{z}$ ,and thus $-a\int \frac{dx}{e-x} = a\int \frac{dz}{z} = a.L.z = a.L.\overline{e-x}$. Whereby $y = \frac{ax}{g} + a.L.\overline{e-x} +$ a constant $Q$. And because with $y$ vanishing, $x$ vanishes also, the constant is found $Q = -a.L.e$,and hence $y = \frac{ax}{g} + a.L.\overline{e-x} - a.L.e = \frac{ax}{g} - a.L.\frac{e}{e-x}$. For indeed $L.e - L.\overline{e-x} = L.\frac{e}{e-x}$ etc.

Note 55: The relation between $DV$ and $Vr$ may be deduced in another way. If indeed on calling $DV = v$ and $Vr = z$ , on account of the similar triangles $DCP$ and $DRV$, $\frac{DP(b)}{DV(v)} = \frac{DC(e)}{DR(x)} = \frac{ev}{b}$ , and thus $e - x = \frac{eb-ev}{b}$ and $\frac{e}{e-x} = \frac{b}{b-v}$ ; similarly there will be $\frac{DC(e)}{CP(f)} = \frac{DE(\frac{ev}{b})}{VR} = \frac{fv}{b}$ and thus $y = Rr = VR - Vr = \frac{fv}{b} - z$. Whereby there will be found $\frac{fv}{b} - z = \frac{aev}{bg} - a.L.\frac{b}{b-v}$ and $z = \frac{fgv-aev}{bg} + a.L.\frac{b}{b-v}$. But $a = \frac{fg}{e-g}$ and thus $ae - ag = fg$ and $fg - ae = -ag$ ; whereby there will be also $z = a.L.\frac{b}{b-v} - \frac{av}{b}$ .

L& J Notes on Corollaries to Proposition IV :

Note 58. For with the times or areas $RDGT$ increasing in an arithmetic progression, the abscissae RC decrease in a geometric progression, and vice versa. Whereby with the verticals $Xr$, which are increasing arithmetically as the areas $RDGT$, the corresponding abscissas $RC$ are decreasing in a geometric progression, and conversely. But on account of the similitude of the triangles D$RX$, $DCZ$, $\frac{DC}{DZ} = \frac{DR}{DX}$ and as $\frac{RC}{ZX}$ on separating: whereby on account of $DC$ and $DZ$ given, $\frac{ZX}{RC}$ is in a given ratio, and thus ZX increases or decreases in the same ratio with $RC$.

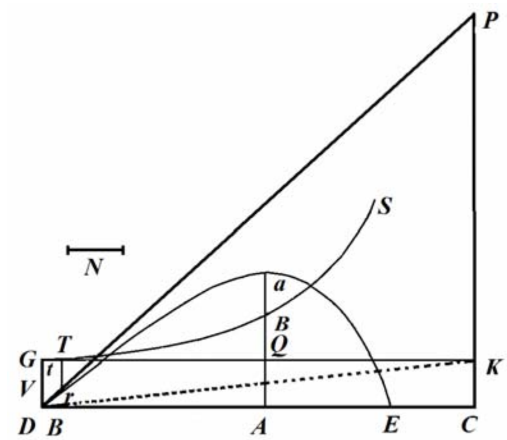
*These may be easily deduced from logarithms* : For the lengths may be called as above $DC = e, CP = f, a = \frac{fg}{e-g}, DR = x, Rr = g$,
$AB = h, Rr = y$,and $y = \frac{ax}{g} - a.L\frac{e}{e-x}$.

On account of the similar triangles $DAY$, $YPZ$, $\frac{DA}{AY} = \frac{YP}{PZ}$ or $\frac{e-g}{f} = \frac{g}{PZ}$ , and hence $PZ = \frac{fg}{e-g} = a$. The similar triangles $DRX$, $YPZ$ also give $\frac{YP(g)}{PZ(a)} = \frac{DR(x)}{RX}$ and $\therefore RX = \frac{ax}{g}$.

From which since there shall be $RC = e - x$ , the equation $y = \frac{ax}{g} - a.L\frac{e}{e-x}$ becomes $Rr = RX - PZ \times L.\frac{DC}{RC}$. Indeed since $PZ \times L.\frac{DC}{RC}$ shall be the logarithm of the ratio $DC$ to $RC$ in the logarithmic curve of which the subtangent is $a$ or $PZ$, by saying as the subtangent of the tables to $PZ$, thus $L.\frac{DC}{RC}$ taken from such to the logarithm of the same quantity in the logarithmic curve of which the subtangent is *PZ*. And thus $PZ \times L.\frac{DC}{RC}$ may be found with the help of tables of common logarithms, and thence the ordinate *Rr* will be obtained to the trajectory *DraF*, and thus any point r in that will be determined.

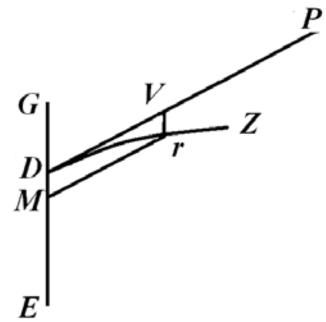
L & J: Note 59. From these the most simple construction of the trajectory by the logarithmic curve is deduced. For, with the same in place which have been prescribed in Corollary 1, with the asymptote *CZ* and with the subtangent *PZ* , the logarithmic curve *DKkG* may be described through the point *D* cutting *RX* in *K*. There may be taken $Xr = RK$ , or $Rr = XK$ , and the point r will be on the trajectory sought *DraF*. For if from the point *K* the perpendicular *KE* is drawn to the *CZ, CE* or *RK* will be the logarithm of the ratio *DC* to *KE* or *RC*, and thus there will be $Rr = RX - RK = XK$ ,and hence $RK = RX - Rr = Xr$. Q. e. d.

Note 60. This construction also has this convenience, that at once the maximum altitude *Aa* may be found and the horizontal distance *DF*. For indeed $Aa = Yk$ ; and if from the point *G* from the intersection of the logarithmic with the line *DZ* there may be sent the perpendicular *GF* to the line *DC*, *DF* will be the amplitude projected, for with *X* coinciding with *G* there is made *XK* or $Rr = 0$ , and thus the point *r* coincides with the point *R* on the horizontal *DC*. Equally the point *r*, by which the trajectory *DraF* may be cut by some right line *Dc* drawn from the point *D* to *CZ*, is found, if *CH* is taken equal to *cZ*, *DH* is drawn cutting the logarithmic in *K*, and from the point *K* the perpendicular *KR* is sent from the point *K* to *DC*, that will cut the line *Dc* in the point sought *r* ; for there will be $\frac{RK}{CH \text{ or } Zc} = \frac{Dr}{Dc} = \frac{Xr}{Zc}$, and thus $Xr = RK$ .




Note 61. Because the velocity of projection is to the terminal velocity, which has been given, as *DP* to *PZ*; if the velocity of projection remains and the line *DP*, the subtangent of the logarithmic *PZ* also remains; and thus one and the same kind of logarithmic suffices for the trajectory *DraF* requiring to be described, however the angle of projection may be changed.

Note 62. For *Vr* is an infinitely small distance that the body will describe by falling under the force of gravity in a medium with a little resistance, and that in the same given time increment it will describe in a medium without resistance. But a body projected in a medium without resistance under the force of gravity will describe the arc of a parabola under the force of gravity *Dr*, the tangent of which is *DP*, the diameter *GDE*, the abscissa $DM = Vr$ , with the ordinate *Mr* equal and parallel to *DV*, and from the nature of the parabola, the rectangle under the latus rectum and the abscissa *DM* or *Vr* is equal to the square of the ordinate *Mr* or *DV*. Whereby the latus rectum of this parabola from the start of the motion is $\frac{DV^2}{Vr}$. Indeed, with *DR* or *Gt* vanishing, the triangle *tGT* becomes $\frac{1}{2}Gt \times Tt = \frac{1}{2}DR \times Tt$, and hence $\frac{tGT}{N} = \frac{DR \times Tt}{2N}$. Again, on account of $KC = DG$ , and the subtangent of the hyperbola equals the abscissa *DC*, from the nature of the hyperbola. But since *GTt* vanishes, there becomes *Tt* to *tG* or *DR* as the ordinate *GD* or *CK* to the subtangent, or to *DC*, and thus $Tt = \frac{CK\times DR}{DC}$. And N will be $\frac{QB\times DC}{CP}$, by construction. Whereby if in place of N and Tt, these values may be substituted into the quantities $\frac{DR\times Tt}{2N} = Vr$, there may be found $\frac{DR^2\times CK\times CP}{2DC^2\times QB}$. *And on account of the proportionalities QB to CK,.....* For *AB* is to *GD* (or *AQ* or *CK*) as *DC* to *AC*, and separating the ratio *QB* is to *CK* as *DK* to *AC*, that is, $\frac{QB}{CK} = \frac{DA}{AC}$.




Notes continued. With the velocity given along the direction of the tangent *DP*, then both a finite distance may be given equally described in a given time in a medium without resistance, as well as from the known effect of gravity in the given time, there is had a finite vertical distance *Vr* described in the same time by the force of gravity, that is, the ordinates and the abscissa of the parabola are given, with which given the latus rectum of that parabola is given.

The curve may be resolved not only by the hyperbola, but also from that logarithmic curve construction. For with *DP* found, the subtangents of the logarithmic curve *PZ* to *DP* must be taken in the ratio of gravity to the resistance of the initial motion; and thus the subtangent *PZ* of the logarithmic also will be as *DP* to 2*DP* to the latus rectum of the parabola.

[L & J: Note 62 :.... *then the resistance of the medium....* that is, by construction, with the ratio of gravity to the whole resistance of the medium at the beginning, the resistance will be given, on account of gravity given; and because $CP\times CA$ is to $DP\times DA$ as $2DP$ to the latus rectum of the parabola, by Cor. 3, that latus rectum will be given.

Note 63 :....... *the velocity from the beginning of the motion is given.......* For with the given latus rectum of the parabola *DrZ*, as well as the weight in the non–resisting medium described, and from the given position of the tangent *DP* with the diameter *DE*, it is possible to describe a parabola; but the velocity may be given at the individual positions of the heavy body describing the given parabola. For let the abscissa be *DM* be equal and parallel to the vertical *Vr*, and with the ordinate *Mr* also equal and parallel to the tangent *DV*; then the velocity is given that the heavy body, falling from the given point *V* through the given distance *Vr*, has at *r*, as well as the time in which it describes that altitude, and hence the time is given in which with a uniform motion the given distance *DV* may be described, and thus the uniform velocity along the tangent *DP* is given, which is the velocity of projection at *D*.

Note 63 : For since the velocity is supposed uniform along the tangent *DV*, if in the give time in which *DV* is described, that velocity may increase, *DV* will increase in the same ratio, with the vertical distance Vr remaining described in this same given time ; but the latus rectum of the parabola *DrZ* is $\frac{DV^2}{Vr}$, by Cor. 3, and the quantity $\frac{DV^2}{Vr}$ with *Vr* fixed, increases as $DV^2$. Whereby the latus rectum of the parabola *DrZ* is increased in the square ratio of the velocity.

The weight may be called G, the initial resistance to the motion R, the latus rectum of the parabola, as above, $\frac{DV^2}{Vr}$, and there will be

$2DP:\frac{DV^2}{Vr}=G:R$, and thus $2DP=\frac{G\times DV^2}{R\times Vr}$, that is , with *Rr* and *G* given, $2DP$ is as $\frac{DV^2}{R}$, and because *R* is as the velocity, or as *DV*, also 2*DP* is as *DV*, or as the velocity, by the above note.

[L & J : Note 64 : *Then by computation...* For with the length and the position *DP* given, *CP* and *DC* are given, and from the given ratio of the resistance at *D* to gravity *DA* and *AC* are given by the construction of this problem itself: But with these given, the curve *DraF* – see the above figures, may be described, and hence the extend of the horizontal motion *DF* may be found from the construction of the hyperbola or by the logarithmic curve (note 59). But if we may wish to perform the calculation of the problem, as we will be able from the equation $y=\frac{ax}{g}-a.L.\frac{e}{e-x}$ (note 63), in which as there shall be $x=DF$, on putting $y=0$, and the equation becomes $\frac{ax}{g}=a.L.\frac{e}{e-x}$, from which by the regression of the series, or by some other approximations, *x* may be found through *g* and *e*, or *DF* through *AC* and *DC*.

L & J : Note 65 : *With the same ratio taken away found by experiment......* and if nothing has been left, the correct ratio has been assumed for the ratio of the resistance to gravity; if there were a certain difference, the difference may be put in place by *MN*. For if the correct ratio were assumed of the resistance to gravity, the curve *DraF* by construction or described by computation is similar to the trajectory that the body will actually describe in the same medium, and hence must be in the same ratio in these curves of homologous lines. For the true trajectory may be determined from the velocity and from the angle of projection to be equal to *PDC* or *pDC*, and from the ratio given of the resistance to gravity; and the curve may be delineated by construction through the assumed length *DP* or *Dp*, which velocity given can always be shown, by the angle *PDC* or *pDC*, and by the ratio of the lines *DA*, *AC* or the ratio of the resistance to gravity, if it were assumed correctly : whereby the whole differences between the true trajectory and the curve described constructed in this manner is in the size of the homologous lines, the ratio of which is the same in either curve. Therefore these curves are similar.

L & J : Note 66 : *And SX will be the true ratio of the resistance to gravity......* For where *MN* or the difference of the ratios $\frac{Ff}{DF}$ which have been found by computation or by experiment, is nothing, the ratio of the resistance to gravity was correct (65). Whereby since *SM* may put that ratio in place, and *MN* may vanish where *SM* becomes *SX*, it is apparent in this case, that the ratio of resistance to gravity to be correctly put in place by the line *SX*. And thus if innumerable abscissas were assumed, and innumerable ordinates were determined by experiment, as curve always touches the point *N*, the accurate ratio of resistance to gravity may be determined by the intersection *X* this with the line *SM*; and thus if many attempts were made, and thus many points *N* were obtained, and by that, and through these the regular curve *NNXN* may be drawn, that nearest the point *X* sought will be determined ; but soon will expound on the method required for drawing a regular curve through a number of given points in the Scholium.

Thus, for the sake of an example, take the ratio of the resistance to gravity as 1 to 10, or $SM = \frac{1}{10}$; with that found there shall be $SX = 2SM = \frac{2}{10} = \frac{1}{5}$; the resistance to gravity will be 1 to 5. From this ratio and with the assumed length DP it is required to deduce the length DF or the amplitude thrown; and because the true ratio of the resistance to gravity found, the trajectory found by calculation or by construction is similar to the trajectory that the body actually describes in the medium, and the amplitude found by calculation *DF* to the amplitude known *DF* by experiment, as the assumed length *DP* to the true length *DP* for the trajectory described in the resisting medium. But with this length found, by Cor. 4, then the curved line *DraF* will be found that the body will indeed itself describe, both with the velocity and resistance of the body at individual places, by Cor. 5. ]